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Error Bounds for Polynomial Product Approximation

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This paper establishes bounds on the uniform error in the approximation of a continuous function defined on a rectangle by polynomial product approximations. The dependence of product approximations on the basis functions used for the associated polynomial spaces is investigated.

1. INTRODUCTION

Several recent papers [1, 6, 7, 9–12, 17, 18] have considered various aspects and extensions of the concept of uniform product approximation. The concern of this paper is the degree of approximation of continuous functions defined on a rectangle by polynomial product approximations.

Let $D = I \times J = [a, b] \times [c, d]$ and $F \in C(D)$, where $C(D)$ denotes the set of continuous real-valued functions on D. Suppose $\{\phi_0, ..., \phi_n\}$ is a Tchebycheff system on I. For $y \in J$ define $F_y \in C(I)$ by $F_y(x) = F(x, y)$ and let

$$B_n(F_y, x) = \sum_{i=0}^{n} f_i(y)\,\phi_i(x) \tag{1.1}$$

be the best approximation of F_y from the linear span Φ_n of $\{\phi_0, ..., \phi_n\}$ in the sense of the uniform norm $\|\cdot\|_I$. The coefficient functions $f_i(y)$, $i = 0, ..., n$,

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are continuous over J (see [17]). Suppose $\{\psi_0, \ldots, \psi_m\}$ is a Tchebycheff system on J, and for $i = 0, \ldots, n$, let

$$S_m(f_i, y) = \sum_{j=0}^{m} f_{ij} \psi_j(y) \tag{1.2}$$

be the best uniform approximation of f_i over J from the linear span Ψ_m of $\{\psi_0, \ldots, \psi_m\}$. The *product approximation* of F over D with respect to the Tchebycheff systems $\{\phi_0, \ldots, \phi_n\}$ and $\{\psi_0, \ldots, \psi_m\}$ is defined to be $P_{n,m}F$, where

$$\begin{aligned}(P_{n,m}F)(x, y) &= \sum_{i=0}^{n} S_m(f_i, y)\, \phi_i(x) \\ &= \sum_{i=0}^{n} \sum_{j=0}^{m} f_{ij} \psi_j(y)\, \phi_i(x).\end{aligned} \tag{1.3}$$

Weinstein [17] has proven the following density theorem for product approximation.

THEOREM 1.1. *Let $\{\phi_i\}_{i=0}^{\infty}$ and $\{\psi_j\}_{j=0}^{\infty}$ be Markoff systems on I and J which are fundmamental in $C(I)$ and $C(J)$, respectively. For $F \in C(D)$, let $P_{n,m}F$ denote the product approximation of F over D with respect to the Tchebycheff systems $\{\phi_0, \ldots, \phi_n\}$ and $\{\psi_0, \ldots, \psi_m\}$. Given $\varepsilon > 0$ there is an $N(\varepsilon)$ and for all $n \geqslant N(\varepsilon)$ there is an $M(\varepsilon, n)$ such that*

$$\|F - P_{n,m}F\|_D = \sup_{(x,y) \in D} |F(x, y) - (P_{n,m}F)(x, y)| < \varepsilon$$

whenever $n \geqslant N(\varepsilon)$ and $m \geqslant M(\varepsilon, n)$.

We are concerned with the case where Φ_n and Ψ_m consist of the algebraic polynomials of degree at most n and m, respectively. In particular, we seek bounds on $\|F - P_{n,m}F\|_D$ which indicate the dependence of $M(\varepsilon, n)$ on n. It will be shown that product approximations are dependent on the basis $\{\phi_0, \ldots, \phi_n\}$ for Φ_n and bounds will be established for product approximations relative to three different bases for the space of polynomials of degree n or less. Two types of error bounds are examined. The first type yields comparisons between $\|F - P_{n,m}F\|_D$ and the degree of approximation

$$E_{n,m}(F) = \inf_{c_{ij}} \left\| F - \sum_{i=0}^{n} \sum_{j=0}^{m} c_{ij} \psi_j \phi_i \right\|_D \tag{1.4}$$

obtained by approximating F on D by polynomials of degree at most n in x and at most m in y. The second type of error bounds are of the Jackson type (see [2]).

It is clear that the degree of approximation of $F \in C(D)$ by product approximations is closely related to smoothness properties of the coefficient functions $f_i(y)$. Although Weinstein [17] proves that the $f_i(y)$ are continuous over J, differentiability properties are not established. In this paper, we give conditions which ensure that the $f_i(y)$, $i = 0,..., n$, are p times continuously differentiable over J. For a particular choice of basis functions for Φ_n, our analysis will produce bounds on df_i/dy, $i = 0,..., n$, which lead to a bound on $\|F - P_{n,m}F\|_D$.

2. Comparisons

In the remainder of this paper, let $I = J = [-1, 1]$, $D = I \times J$, and Φ_n and Ψ_m be the sets of algebraic polynomials of degree at most n and m, respectively. We consider product approximations relative to the following three bases for Φ_n: $\{1, x,..., x^n\}$, $\{T_0(x), T_1(x),..., T_n(x)\}$, where $T_i(x)$ is the ith degree Tchebycheff polynomial, and $\{l_0(x), l_1(x),..., l_n(x)\}$, where the $l_i(x)$ are the Lagrange polynomials with nodes

$$\xi_k = \mathrm{Cos}(2k + 1)\pi/(2n + 2), \qquad k = 0, 1,..., n, \tag{2.1}$$

the zeros of the $(n + 1)$st degree Tchebycheff polynomial. For $F \in C(D)$ and $y \in J$, let

$$\begin{aligned} B_n(F_y, x) &= \sum_{i=0}^{n} f_i(y)\, x^i \\ &= \sum_{i=0}^{n} f_i^T(y)\, T_i(x) \\ &= \sum_{i=0}^{n} f_i^l(y)\, l_i(x) \end{aligned} \tag{2.2}$$

be the best uniform approximation of F_y over I from Φ_n. The respective product approximations of F shall be denoted by $P_{n,m}F$, $P^T_{n,m}F$, and $P^l_{n,m}F$. For definitions and properties of the Tchebycheff and Lagrange polynomials, see Davis [3]. The following example indicates that $P_{n,m}F$, $P^T_{n,m}F$, and $P^l_{n,m}F$ may differ.

Example. Let $n = 2$, $m = 1$, and

$$\begin{aligned} F(x, y) &= -4y(1 + y), \qquad -1 \leqslant y < 0 \\ &= 8x^2y(1 - y), \qquad 0 \leqslant y \leqslant 1. \end{aligned}$$

Since each $F_y \in \Phi_2 = \text{span}\{1, x, x^2\}$,

$$B_2(F_y, x) = f_0(y) + f_2(y)\, x^2,$$

where

$$\begin{aligned} f_0(y) &= -4y(1+y), & -1 \leqslant y < 0 \\ &= 0, & 0 \leqslant y \leqslant 1 \end{aligned}$$

and

$$\begin{aligned} f_2(y) &= 0, & -1 \leqslant y < 0 \\ &= 8y(1-y), & 0 \leqslant y \leqslant 1. \end{aligned}$$

Application of the alternation theorem [2, p. 75] yields $S_1(f_0, y) = 1/2$ and $S_1(f_2, y) = 1$. Thus

$$(P_{2,1}F)(x, y) = \tfrac{1}{2} + x^2.$$

Converting $B_2(F_y, x)$ to Tchebycheff and Lagrange polynomials gives

$$\begin{aligned} B_2(F_y, x) &= [f_0(y) + \tfrac{1}{2} f_2(y)]\, T_0(x) + \tfrac{1}{2} f_2(y)\, T_2(x) \\ &= [f_0(y) + \tfrac{3}{4} f_2(y)]\, l_0(x) + f_0(y)\, l_1(x) \\ &\quad + [f_0(y) + \tfrac{3}{4} f_2(y)]\, l_2(x), \end{aligned}$$

where the nodes for the $l_i(x)$, $i = 0, 1, 2$, are $\sqrt{3}/2$, 0, and $-\sqrt{3}/2$. Best approximation of the coefficient functions yields

$$(P^T_{2,1}F)(x, y) = \tfrac{1}{2} T_0(x) + \tfrac{1}{2} T_2(x) = x^2$$

and

$$(P^l_{2,1}F)(x, y) = \tfrac{3}{4} l_0(x) + \tfrac{1}{2} l_1(x) + \tfrac{3}{4} l_2(x) = \tfrac{1}{3} x^2 + \tfrac{1}{2}.$$

Thus $P_{2,1}F$, $P^T_{2,1}F$, and $P^l_{2,1}F$ may differ. As expected, the uniform errors $\|F - P_{2,1}F\|_D = 3/2$, $\|F - P^T_{2,1}F\|_D = 1$, and $\|F - P^l_{2,1}F\|_D = 7/6$ also differ.

It is evident that this basis dependence results from the nonlinearity of the Tchebycheff approximation operator S_m. Since S_m does not depend on the polynomial basis for Ψ_m, the product approximation of $F \in C(D)$ is independent of the choice of basis for Ψ_m once a basis for Φ_n is fixed.

The first theorem of this section produces a comparison between $\|F - P^l_{n,m}F\|_D$ and the error $E_{n,m}(F)$ defined by (1.4).

THEOREM 2.1. *If $F \in C(D)$, then*

$$\|F - P^l_{n,m}F\|_D \leqslant \left[3 + \frac{4}{\pi} \ln(n+1)\right] E_{n,m}(F). \tag{2.3}$$

Proof. Select $(\hat{P}_{n,m}F)(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{m} b_{ij}\psi_j(y)\, l_i(x)$ such that $\|F - \hat{P}_{n,m}\|_D = E_{n,m}(F)$. It is easy to see that $|F(x, y) - \sum_{i=0}^{n} f_i^l(y)\, l_i(x)| \leqslant E_{n,m}(F)$ for each $(x, y) \in D$. Since $l_i(\xi_k) = 1$ if $k = i$ and $l_i(\xi_k) = 0$ if $k \neq i$, $f_i^l(y) = B_n(F_y, \xi_i)$, and $\sum_{j=0}^{m} b_{ij}\psi_j(y) = (\hat{P}_{n,m}F)(\xi_i, y)$ for $i = 0,..., n$ and $y \in J$. Thus

$$\begin{aligned}\left|f_i^l(y) - \sum_{j=0}^{m} b_{ij}\psi_j(y)\right| &= |B_n(F_y, \xi_i) - (\hat{P}_{n,m}F)(\xi_i, y)| \\ &\leqslant |F_y(\xi_i) - B_n(F_y, \xi_i)| \\ &\quad + |F(\xi_i, y) - (\hat{P}_{n,m}F)(\xi_i, y)|. \\ &\leqslant 2E_{n,m}(F).\end{aligned}$$

Therefore $\|f_i^l - \sum_{j=0}^{m} b_{ij}\psi_j\|_J \leqslant 2E_{n,m}(F)$, and hence $\|f_i^l - S_m(f_i^l, \cdot)\|_J \leqslant 2E_{n,m}(F)$. Now for $(x, y) \in D$,

$$\begin{aligned}|F(x, y) - (P_{n,m}^l F)(x, y)| &\leqslant \left|F(x, y) - \sum_{i=0}^{n} f_i^l(y)\, l_i(x)\right| \\ &\quad + \sum_{i=0}^{n} |f_i^l(y) - S_m(f_i^l, y)|\, |l_i(x)| \\ &\leqslant \left(1 + 2\sum_{i=0}^{n} |l_i(x)|\right) E_{n,m}(F). \qquad (2.4)\end{aligned}$$

From Rivlin [15, p. 13], $\sum_{i=0}^{n} |l_i(x)| \leqslant 1 + (2/\pi)\log(n + 1)$. This observation and (2.4) now imply (2.3).

Remark. We note that $\|F - P_{n,m}^l F\|_D = 0(\log n)\, E_{n,m}(F)$ and that the coefficient on the right-hand side of (2.3) is independent of m. This is a significant theoretical improvement over a corresponding result for tensor product interpolation using the zeros of $T_{n+1}(x)$ and $T_{m+1}(y)$ for the nodes (see deBoor [4]). In this later case the error bound over D is $0(\log n \log m)\, E_{n,m}(F)$.

If the ϕ_i, $i = 0,..., n$, are orthogonal polynomials, results similar to (2.3) can be obtained. In particular, we apply orthogonality properties of the Tchebycheff polynomials to derive an error bound corresponding to (2.3).

THEOREM 2.2. *If $F \in C(D)$, then*

$$\|F - P_{n,m}^T F\|_D \leqslant (3 + 2n\sqrt{2})\, E_{n,m}(F). \qquad (2.5)$$

Proof. Let $\hat{P}_{n,m}F$ be as in the proof of Theorem 2.1. Then $\hat{P}_{n,m}$ may be written as $(\hat{P}_{n,m}F)(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{m} \hat{b}_{ij} \psi_j(y) T_i(x)$. For $i = 0, \dots, n$, and $y \in J$ an application of the Cauchy–Schwarz inequality yields

$$\begin{aligned} &\left| f_i^T(y) - \sum_{j=0}^{m} \hat{b}_{ij} \psi_j(y) \right| \\ &\quad = \frac{1}{\|T_i\|_2^2} \left| \int_{-1}^{1} [B_n(F_y, x) - (\hat{P}_{n,m}F)(x, y)]\, T_i(x)(1 - x^2)^{-1/2}\, dx \right| \\ &\quad \leqslant \frac{1}{\|T_i\|_2^2} \left\{ \int_{-1}^{1} [B_n(F_y, x) - (\hat{P}_{n,m}F)(x, y)]^2 (1 - x^2)^{-1/2}\, dx \right\}^{1/2} \\ &\qquad \cdot \left\{ \int_{-1}^{1} T_i^2(x)(1 - x^2)^{-1/2}\, dx \right\}^{1/2}, \end{aligned}$$

where $\|T_i\|_2^2 = \int_{-1}^{1} T_i^2(x)(1 - x^2)^{-1/2}\, dx$. Thus

$$\left| f_i^T(y) - \sum_{j=0}^{m} \hat{b}_{ij} \psi_j(y) \right| \leqslant \frac{\sqrt{\pi}}{\|T_i\|_2} \sup_{x \in I} |B_n(F_y, x) - (\hat{P}_{n,m}F)(x, y)|.$$

As in the proof of Theorem 2.1, this inequality implies that

$$\left| f_i^T(y) - \sum_{j=0}^{m} \hat{b}_{ij} \psi_j(y) \right| \leqslant \frac{2\sqrt{\pi}}{\|T_i\|_2} E_{n,m}(F).$$

Thus $\|f_i^T - S_m(f_i^T, \cdot)\|_J \leqslant 2\sqrt{\pi}\, E_{n,m}(F)/\|T_i\|_2$. For $(x, y) \in D$, the argument given to obtain (2.4) now implies that

$$\begin{aligned} &|F(x, y) - (P_{n,m}^T F)(x, y)| \\ &\quad \leqslant \left(1 + 2\sqrt{\pi} \sum_{i=0}^{n} |T_i(x)|/\|T_i\|_2\right) E_{n,m}(F) \\ &\quad \leqslant (3 + 2n\sqrt{2})\, E_{n,m}(F). \end{aligned}$$

The error bound (2.5) is not as strong as that given by (2.3). However, in certain secial cases, (2.5) can be significantly improved.

THEOREM 2.3. *Suppose that $F(x, y) = f(x)\, g(y) + h(y)$, where $f \in C(I)$ and $g, h \in C(J)$. Then*

$$\|F - P_{n,m}^T F\|_D \leqslant M E_{n,m}(F), \tag{2.6}$$

where M is a positive constant independent of n and m.

Proof. Since $F(x, y) = f(x)\, g(y) + h(y)$,

$$B_n(F_y, x) = \{B_n(f, x)\}\, g(y) + h(y) \tag{2.7}$$

where $B_n(f, x) = \sum_{i=0}^{n} a_i T_i(x)$. Equation (2.7) implies that $f_0^T(y) = a_0 g(y) + h(y)$, and that $f_i^T(y) = a_i g(y)$, $i = 1,..., n$. From the proof of Theorem 2.2 we have that

$$\|f_i^T(y) - S_m(f_i^T, \cdot)\|_J \leqslant 2\sqrt{\pi}\, E_{n,m}(F)/\|T_i\|_2, \qquad i = 0, 1,..., n. \tag{2.8}$$

For $i = 1,..., n$, this inequality implies that

$$\max_{1 \leqslant k \leqslant n} |a_k| \left| g(y) - \sum_{i=0}^{n} c_j \psi_j(y) \right| \leqslant 2\sqrt{\pi}\, E_{n,m}(F)/\|T_i\|_2, \tag{2.9}$$

where $\sum_{j=0}^{m} c_j \psi_j(y)$ is the best uniform approximation to g on J. For $(x, y) \in D$,

$$\begin{aligned} |(P_{n,m}^T F)(x, y) - B_n(F_y, x)| &= \left| \sum_{i=0}^{n} \sum_{j=0}^{m} a_{ij} \psi_j(y)\, T_i(x) - \sum_{i=0}^{n} f_i^T(y)\, T_i(x) \right| \\ &= \left| \sum_{i=1}^{n} a_i \left[g(y) - \sum_{j=0}^{m} c_i \psi_j(y) \right] T_i(x) \right. \\ &\quad \left. + [a_0 g(y) + h(y) - S_m(f_0^T, y)]\, T_0(x) \right|. \end{aligned}$$

Therefore (2.8) implies that

$$\begin{aligned} |(P_{n,m}^T F)(x, y) - B_n(F_y, x)| &\leqslant \left| \sum_{i=1}^{n} a_i T_i(x) \right| \left| g(y) - \sum_{j=0}^{m} c_j \psi_j(y) \right| \\ &\quad + 2\sqrt{\pi}\, E_{n,m}(F)/\|T_0\|_2 \\ &= \max_{1 \leqslant k \leqslant n} |a_k| \left| g(y) - \sum_{j=0}^{m} c_j \psi_j(y) \right| \left| \sum_{i=1}^{n} a_i T_i(x) \right| \Big/ \max_{1 \leqslant k \leqslant n} |a_k| \\ &\quad + 2E_{n,m}(F). \end{aligned}$$

Applying (2.9) to this inequality results in

$$\begin{aligned} |(P_{n,m}^T F)(x, y) - B_n(F_y, x)| &\leqslant 2\sqrt{\pi}\, E_{n,m}(F) \left| \sum_{i=1}^{n} a_i T_i(x) \right| \Big/ \max_{1 \leqslant k \leqslant n} |a_k| + 2E_{n,m}(F). \end{aligned} \tag{2.10}$$

Clearly

$$\left\| \sum_{i=1}^{n} a_i T_i \right\|_I \leqslant 2\, \|f\|_I + |a_0|. \tag{2.11}$$

But

$$\begin{aligned} |a_0| &= \left| \frac{1}{\pi} \int_{-1}^{1} B_n(f, x)\, T_0(x)(1-x^2)^{-1/2}\, dx \right| \\ &\leqslant 2\, \|f\|_I. \end{aligned}$$

Thus (2.11) implies that

$$\left\| \sum_{i=1}^{n} a_i T_i \right\|_I \leqslant 4\, \|f\|_I. \tag{2.12}$$

Also for $j = 1,..., n$,

$$\begin{aligned} |a_j| &= \left| \frac{2}{\pi} \int_{-1}^{1} B_n(f, x)\, T_j(x)(1-x^2)^{-1/2}\, dx \right| \\ &= \left| \frac{2}{\pi} \int_{-1}^{1} f(x)\, T_j(x)(1-x^2)^{-1/2}\, dx \right. \\ &\qquad \left. - \frac{2}{\pi} \int_{-1}^{1} \{f(x) - B_n(f, x)\}\, T_j(x)(1-x^2)^{-1/2}\, dx \right| \\ &\geqslant \frac{2}{\pi} \left| \int_{-1}^{1} f(x)\, T_j(x)(1-x^2)^{-1/2}\, dx \right| \\ &\qquad - \frac{2}{\pi} \|f - B_n(f, \cdot)\|_I \int_{-1}^{1} |T_j(x)|\, (1-x^2)^{-1/2}\, dx. \end{aligned} \tag{2.13}$$

The expression $(2/\pi) \int_{-1}^{1} f(x)\, T_j(x)(1-x^2)^{-1/2}\, dx$ represents the $(j+1)$st Fourier–Tchebycheff coefficient of F. If f is not a constant function (in which case F is a function of a single variable and the theorem is trivial), then there is a $j^* \geqslant 1$ such that

$$\frac{2}{\pi} \left| \int_{-1}^{1} f(x)\, T_{j^*}(x)(1-x^2)^{-1/2}\, dx \right| = 2\alpha > 0.$$

This inequality and (2.13) now imply that there is an $N \geqslant j^*$ such that for all $n \geqslant N$,

$$\max_{1 \leqslant k \leqslant n} |a_k| \geqslant \alpha > 0, \tag{2.14}$$

where α is independent of n. Combining (2.10), (2.12), and (2.14) results in

$$|(P_{n,m}^T F)(x, y) - B_n(F_y, x)| \leqslant (2 + 8\sqrt{2}\,\|f\|_I/\alpha)\, E_{n,m}(F).$$

This inequality now implies that

$$|(P_{n,m}^T F)(x, y) - F(x, y)| \leqslant \{(3 + 8\sqrt{2}\,\|f\|_I/\alpha)\}\, E_{n,m}(F).$$

for all $n \geqslant N$.

Remark. Theoretically speaking, Theorem 2.3 establishes for a common class of functions $F \in C(D)$ that uniform product approximation with appropriate basis functions yields error bounds proportional to the best possible error bound (in the sense of (1.4)). In contrast, there exist functions of the type given in Theorem 2.3 for which the norm of the tensor product interpolant (using the zeros of $T_{n+1}(x)$ and $T_{m+1}(y)$ for the nodes) diverges as $n + m \to +\infty$.

3. Jackson Type Estimates for $P_{n,m}F$

In this section, we determine conditions on F which ensure that the coefficient functions in (2.2) have continuous pth derivatives. This analysis leads to bounds on the derivatives of the coefficient functions $f_i(y)$. From this we shall derive Jackson type bounds for $\|F - P_{n,m}F\|_D$.

We assume that $F \in C(D)$ and $\partial^{n+1}F/\partial x^{n+1}$ exists and is non-zero for all $(x, y) \in (-1, 1) \times J$. In this case, $F_y - B_n(F_y, \cdot)$ has a unique alternation set

$$-1 = x_0(y) < x_1(y) < \cdots < x_{n+1}(y) = 1$$

on which the function $F_y - B_n(F_y, \cdot)$ attains the values $\pm\|F_y - B_n(F_y, \cdot)\|_I$ with alternating signs (see [16]). Let

$$\lambda(y) = F_y(x_0(y)) - B_n(F_y, x_0(y)).$$

Then a vector $(\gamma, a_0,..., a_n, \xi_1,..., \xi_n)$ in the open subset $\mathcal{O} = \{(\gamma, a_0,..., a_n, \xi_1,..., \xi_n)$: γ, $a_0,..., a_n \in R$, $-1 < \xi_1 < \cdots < \xi_n < 1\}$ of R_{2n+2} satisfies the conditions

(i) $B_n(F_y, x) = \sum_{i=0}^n a_i x^i$,

(ii) $-1 = \xi_0 < \xi_1 < \cdots < \xi_n < \xi_{n+1} = 1$ constitutes an alternation set for $F_y - B_n(F_y, \cdot)$, and

(iii) $\gamma = \lambda(y)$

if and only if

$$F(\xi_k, y) - \sum_{i=0}^{n} a_i \xi_k^i = (-1)^i \gamma, \qquad k = 0,..., n+1, \tag{3.1}$$

and

$$\left.\frac{\partial F}{\partial x}\right|_{(\xi_l, y)} - \sum_{i=1}^{n} a_i i \xi_l^{i-1} = 0, \qquad l = 1,..., n. \tag{3.2}$$

The necessity of the system (3.1) and (3.2) follows from the alternation theorem. The sufficiency of (3.1) and (3.2) can be established by a Rolle's theorem argument.

THEOREM 3.1. *If F, $\partial F/\partial x \in C^p(D)$ and $\partial^{n+1}F/\partial x^{n+1}$ exists and is nonzero for all $(x, y) \in (-1, 1) \times J$, then the functions $f_i(y)$, $f_i^T(y)$, and $f_i^!(y)$, defined in (2.2), possess continuous pth derivatives over J.*

Proof. We note that the system (3.1) and (3.2) of $2n+2$ equations in the $2n+2$ variables γ, $a_0,..., a_n$, $\xi_1,..., \xi_n$ has a unique solution in $\mathcal{O}$ for each $y \in J$, namely, $\gamma = \lambda(y)$, $a_i = f_i(y)$, $i = 0,..., n$, $\xi_k = x_k(y)$, $k = 1,..., n$. We rewrite the system (3.1) and (3.2) as

$$H_k(\gamma, a_0,..., a_n, \xi_1,..., \xi_n)$$
$$= (-1)^{k+1}\gamma + F(\xi_k, y) - \sum_{i=0}^{n} a_i \xi_k^i = 0, \qquad k = 0,..., n+1, \tag{3.1}$$
$$G_l(\gamma, a_0,..., a_n, \xi_1,..., \xi_n)$$
$$= \left.\frac{\partial F}{\partial x}\right|_{(\xi_l, y)} - \sum_{i=1}^{n} a_i i \xi_l^{i-1} = 0, \qquad l = 1,..., n. \tag{3.2}$$

Since F, $\partial F/\partial x \in C^p(D)$, each H_k and G_l is of class C^p on $\mathcal{O} \times J$. In view of the Implicit Function Theorem [13, p. 210], the functions $\lambda(y)$, $f_i(y)$, $i = 0,..., n$, and $x_k(y)$, $k = 1,..., n$, will be of class C^p on $(-1, 1)$ if the Jacobian

$$\Delta(y) = \left.\frac{\partial(H_0,..., H_{n+1}, G_1,..., G_n)}{\partial(\gamma, a_0,..., a_n, \xi_1,..., \xi_n)}\right|_{(\lambda(y), f_0(y),..., f_n(y), x_1(y),..., x_n(y))} \neq 0$$

for all $y \in (-1, 1)$. In a fashion similar to that in Nitsche [14], we have that

$$\Delta(y) = \sigma \prod_{k=1}^{n} \left(\left.\frac{\partial^2 F}{\partial x^2}\right|_{(x_k(y), y)} - \sum_{i=2}^{n} f_i(y)\, i(i-1)\, x_k^{i-2}(y) \right) \sum_{j=0}^{n+1} D_j(y),$$

where $\sigma = \pm 1$ and where

$$D_j(y) = \begin{vmatrix} 1 & x_0(y) & \cdots & x_0^n(y) \\ \vdots & \vdots & & \vdots \\ 1 & x_{j-1}(y) & \cdots & x_{j-1}^n(y) \\ 1 & x_{j+1}(y) & \cdots & x_{j+1}^n(y) \\ \vdots & \vdots & & \vdots \\ 1 & x_{n+1}(y) & \cdots & x_{n+1}^n(y) \end{vmatrix}.$$

The $D_j(y)$ are non-zero and have the same sign (see [2, p. 74]), and thus $\sum_{j=0}^{n+1} D_j(y) \neq 0$. A straight forward Rolle's theorem argument ensures that each

$$\frac{\partial^2 F}{\partial x^2}\bigg|_{(x_k(y),y)} - \sum_{i=2}^{n} f_i(y)\, i(i-1)\, x_k^{i-2}(y) \neq 0$$

Thus $\Delta(y) \neq 0$. Therefore each $f_i(y)$ is of class C^p on $(-1, 1)$. By extending F so that F and $\partial F/\partial x$ are of class C^p on $I \times (-1-\varepsilon, 1+\varepsilon)$ for some $\varepsilon > 0$, the above argument can be used to see that each $f_i(y)$ is of class C^p on J.

We complete the proof of Theorem 3.1 by noting that the $f_i^T(y)$ and $f_i^I(y)$ are linear combinations of the $f_i(y)$ and thus are also of class C^p on J.

Remark. The above proof also establishes that $\lambda(y)$ and the $x_k(y)$ possess continuous pth derivatives whenever F satisfies the conditions of Theorem 3.1. In this regard, Weinstein [17] proved the continuity of the $x_k(y)$ when $\partial^{n+1}F/\partial x^{n+1}$ is non-vanishing. The continuous variance of the extremal points with respect to y facilitates the computing of product approximations (see [12, 17]).

The following lemma will be useful in establishing bounds on the uniform error of approximation.

LEMMA 3.2. *Let $F \in C(D)$. Then*

$$\|F - P_{n,m}F\|_D \leqslant \sup_{y \in J} \|F_y - B_n(F_y, \cdot)\|_I + \sum_{i=0}^{n} \omega(f_i, J, \pi/(m+1)),$$

where $\omega(g, J, \delta)$ denotes the modulus of continuity of g [2, p. 86].

Proof. For $(x, y) \in D$,

$$\begin{aligned} &|F(x, y) - (P_{n,m}F)(x, y)| \\ &\quad \leqslant |F_y(x) - B_n(F_y, x)| + |B_n(F_y, x) - (P_{n,m}F)(x, y)| \\ &\quad \leqslant \sup_{y \in J} \|F_y - B_n(F_y, \cdot)\|_I + \sum_{i=0}^{n} |f_i(y) - S_m(f_i, y)|\,|x^i| \\ &\quad \leqslant \sup_{y \in J} \|F_y - B_n(F_y, \cdot)\|_I + \sum_{i=0}^{n} \|f_i - S_m(f_i, \cdot)\|_J \\ &\quad \leqslant \sup_{y \in J} \|F_y - B_n(F_y, \cdot)\|_I + \sum_{i=0}^{n} \omega(f_i, J, \pi/(m+1)), \end{aligned}$$

where the last inequality follows from Jackson's theorem [2, p. 147].

We now establish a bound on the uniform error $\|F - P_{n,m}F\|_D$. We first estimate the derivatives $f_i^{(k)}(y) = d^k f_i/dy^k$, $k = 0,\ldots, n$. Equations (3.1) and (3.2) may be restated as

$$F(x_k(y), y) - \sum_{i=0}^{n} f_i(y)\, x_k^i(y) = (-1)^k \lambda(y), \qquad i = 0,\ldots, n+1, \quad (3.3)$$

and

$$\left.\frac{\partial F}{\partial x}\right|_{(x_k(y), y)} - \sum_{i=1}^{n} f_i(y)\, i x_k^{i-1}(y) = 0, \qquad i = 1,\ldots, n. \quad (3.4)$$

If F, $\partial F/\partial x \in C'(D)$ and $\partial^{n+1}F/\partial x^{n+1}$ exists and is non-vanishing on $(-1, 1) \times J$, then we may differentiate the identity (3.3) with respect to y, and using (3.4) or the fact that $x_0(y) = -1$ and $x_{n+1}(y) = 1$, we obtain

$$\begin{aligned} &\left.\frac{\partial F}{\partial y}\right|_{(x_k(y), y)} - \sum_{i=0}^{n} f_i'(y)\, x_k^i(y) \\ &\qquad = (-1)^k \lambda'(y), \qquad k = 0,\ldots, n+1. \end{aligned} \quad (3.5)$$

Let $H_y(x) = \sum_{i=0}^{n} f_i'(y)\, x^i$. Then

$$f_i'(y) = \frac{1}{i!} H_y^{(i)}(0). \quad (3.6)$$

We now use (3.5) and (3.6) to estimate $f_i'(y)$ and thus the moduli of continuity of $f_i(y)$.

LEMMA 3.3. *Suppose* F, $\partial F/\partial x \in C'(D)$, $\partial^{j+1}F/\partial x^j\, \partial y \in C(D)$, $j =$

$0,\dots, n+1$, *and* $\partial^{n+1}F/\partial x^{n+1}$ *exists and does not vanish for all* $(x, y) \in (-1, 1) \times J$. *Then*

$$\|f_i'\|_J \leqslant \frac{1}{i!\,(n-i+1)!}\left\|\frac{\partial^{n+2}F}{\partial x^{n+1}\,\partial y}\right\|_D + \frac{1}{i!}\sup_{y\in J}\left|\left.\frac{\partial^{i+1}F}{\partial x^i\,\partial y}\right|_{(0,y)}\right|, \qquad i=0,\dots,n. \tag{3.7}$$

Proof. In the following argument, we fix y. Equation (3.5) and repeated applications of Rolle's theorem imply that the polynomial $H_y^{(i)}(x)$ of degree $n-i$ or less interpolates the function $\partial^{i+1}F/\partial x^i\,\partial y$ at $n-i+1$ points (or more) in $(-1, 1)$, $i=0,\dots, n$. By the well-known error formula for Lagrange interpolation [2, p. 60],

$$\left.\frac{\partial^{i+1}F}{\partial x^i\,\partial y}\right|_{(x,y)} - H_y^{(i)}(x) = \frac{1}{(n-i+1)!}\left.\frac{\partial^{n+2}F}{\partial x^{n+1}\,\partial y}\right|_{(\xi,y)} \prod_{j=0}^{n-i}(x-x_j^*),$$

where the x_j^* are the points of interpolation and $\xi \in (-1, 1)$ depends on x. For $x=0$, we obtain

$$|H_y^{(i)}(0)| \leqslant \frac{1}{(n-i+1)!}\left\|\frac{\partial^{n+2}F}{\partial x^{n+1}\,\partial y}\right\|_D + \left|\left.\frac{\partial^{i+1}F}{\partial x^i\,\partial y}\right|_{(0,y)}\right|.$$

Equation (3.6) now implies that

$$|f_i'(y)| \leqslant \frac{1}{i!\,(n-i+1)!}\left\|\frac{\partial^{n+2}F}{\partial x^{n+1}\,\partial y}\right\|_D + \frac{1}{i!}\left|\left.\frac{\partial^{i+1}F}{\partial x^i\,\partial y}\right|_{(0,y)}\right|.$$

Thus Lemma 3.3 is proven.

If $f_i' \in C(J)$, then the mean value theorem implies that $\omega(f_i, J, \delta) \leqslant \delta\,\|f_i'\|_J$. Thus Lemma 3.3 provides an estimate for $\omega(f_i, J, \delta)$.

THEOREM 3.4. *Suppose* F, $\partial F/\partial x \in C'(D)$, $\partial^{j+1}F/\partial x^j\,\partial y \in C(D)$, $j=0,\dots, n+1$, *and* $\partial^{n+1}F/\partial x^{n+1}$ *exists and is non-vanishing for all* $(x, y) \in (-1, 1) \times J$. *Then if* $n \geqslant 2$,

$$\|F - P_{n,m}F\|_D \leqslant \frac{\pi^2}{4n(n+1)}\left\|\frac{\partial^2 F}{\partial x^2}\right\|_D + \frac{\pi}{(m+1)}\left\{\frac{2^{n+1}-1}{(n+1)!}\left\|\frac{\partial^{n+2}F}{\partial x^{n+1}\,\partial y}\right\|_D + \sum_{i=0}^{n}\frac{1}{i!}\sup_{y\in J}\left|\left.\frac{\partial^{i+1}F}{\partial x^i\,\partial y}\right|_{(0,y)}\right|\right\}. \tag{3.6}$$

Proof. Since $\partial F/\partial x \in C'(D)$, $\partial^2 F/\partial x^2 \in C(D)$, then by Jackson's theorem [2, p. 147]

$$\sup_{y\in J} \|F_y - B_n(F_y, \cdot)\|_I \leqslant \frac{\pi^2}{4n(n+1)} \left\| \frac{\partial^2 F}{\partial x^2} \right\|_D.$$

Applying Lemmas 3.2 and 3.3, we have

$$\begin{aligned}
\|F - P_{n,m}F\|_D &\leqslant \frac{\pi^2}{4n(n+1)} \left\| \frac{\partial^2 F}{\partial x^2} \right\|_D \\
&\quad + \frac{\pi}{(m+1)} \sum_{i=0}^{n} \left\{ \frac{1}{i!\,(n-i+1)!} \left\| \frac{\partial^{n+2} F}{\partial x^{n+1}\,\partial y} \right\|_D \right. \\
&\quad \left. + \frac{1}{i!} \sup_{y\in J} \left| \left. \frac{\partial^{i+1} F}{\partial x^i\,\partial y} \right|_{(0,y)} \right| \right\} \\
&= \frac{\pi^2}{4n(n+1)} \left\| \frac{\partial^2 F}{\partial x^2} \right\|_D \\
&\quad + \frac{\pi}{(m+1)} \left\{ \left\| \frac{\partial^{n+2} F}{\partial x^{n+1}\,\partial y} \right\|_D \sum_{i=0}^{n} \frac{1}{i!\,(n-i+1)!} \right. \\
&\quad \left. + \sum_{i=0}^{n} \frac{1}{i!} \sup_{y\in J} \left| \left. \frac{\partial^{i+1} F}{\partial x^i\,\partial y} \right|_{(0,y)} \right| \right\}.
\end{aligned}$$

The proof is completed by observing that

$$\sum_{i=0}^{n} \frac{1}{i!\,(n-i+1)!} = \frac{1}{(n+1)!} \sum_{i=0}^{n} \binom{n+1}{i} = \frac{2^{n+1}-1}{(n+1)!}.$$

Relative to Theorem 1.1, the following corollary shows that the $M(\varepsilon, n)$ does not depend on n whenever F satisfies the conditions of Theorem 3.4 for all n and

$$\left\| \frac{\partial^{n+1} F}{\partial x^n\,\partial y} \right\|_D \leqslant An!/2^n.$$

COROLLARY 3.5. *Suppose that F, $\partial F/\partial x \in C'(D)$, $\partial^{j+1} F/\partial x^j\,\partial y \in C(D)$, $j = 0, 1,\dots$, and that $\partial^j F/\partial x^j$ exists and is non-zero for all $(x, y) \in (-1, 1) \times J$, $j = 3, 4,\dots$. Also suppose there is a positive constant A such that*

$$\left\| \frac{\partial^{j+1} F}{\partial x^j\,\partial y} \right\|_D \leqslant Aj!/2^j,$$

$j = 0, 1, \ldots$. *Then if* $n \geqslant 2$

$$\|F - P_{n,m}F\|_D \leqslant \frac{\pi^2}{4n(n+1)} \left\| \frac{\partial^2 F}{\partial x^2} \right\|_D + \frac{3A\pi}{m+1}. \tag{3.8}$$

The bound (3.8) implies that $\|F - P_{n,m}F\|_D$ can be made small by independently choosing n and m large. In this sense, the result of Corollary 3.5 is stronger than that of Theorem 1.1. We further remark that the bound of (3.8) in the case $m = n$ is comparable to the Jackson estimate for multidimensional best approximation (see Feinerman and Newman [4, p. 101]).

The conditions of Corollary 3.5 are satisfied by such functions as $\exp[x(y+p)]$, where $|p| > 1$, $(x + y + p)^{1/\mu}$, where $\mu \geqslant 2$ and $p > 4$, and others of similar construction. The functions $\sin((x+y)/p)$ and $\cos((x+y)/p)$, where $p \geqslant 4/\pi$ possess non-vanishing $(n+1)$st partial derivatives with respect to x for alternating values of n, in which case the bound (3.8) holds for these values of n.

The techniques of estimating the $f_i'(y)$ do not produce bounds on the higher derivatives of the $f_i(y)$, since neither (3.4) nor derivatives of (3.4) can be utilized to reduce the corresponding differentiated equations in (3.5).

4. Conclusions

In the preceding sections, smoothness properties and error bounds have been examined and established for certain uniform product approximations. These theorems extend the continuity and density theorems of Weinstein [17]. The results of Section 2 suggest that product approximations are strongly dependent on the choice of basis functions. Indeed, Theorems 2.1 and 2.3 demonstrate that for appropriate choices of basis functions uniform product approximations are nearly as good as best approximations. In addition, the error bounds of Theorems 2.1 and 2.3 are sharper than that for tensor product interpolation [4]. The error estimates of Section 3 show that for certain $F \in C(D)$ the error is $O(1/n^2) + O(1/m)$. Thus the dependence of $M(\varepsilon, n)$ on n in Theorem 1.1 is eliminated.

The authors feel that the differentiability properties in Section 3 for $f_i(y)$, $i = 0, \ldots, n$, have not been fully exploited. Different techniques of proof may more completely utilize the pth order differentiability of $f_i(y)$, $i = 0, \ldots, n$. Further investigations in this direction are appropriate.

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